APPENDIX A
MATHEMATICAL ADDENDUM

1 INTRODUCTION

The purpose of this appendix is to provide the reader with a ready reference to some mathematical results that are used in the book. This appendix is divided into two main sections: The first, Sec. 2 below, gives results that are, for the most part, combinatorial in nature, and the last gives results from calculus. No attempt is made to prove these results, although sometimes a method of proof is indicated.

2 NONCALCULUS

2.1 Summation and Product Notation

A sum of terms such as $n_3 + n_4 + n_5 + n_6 + n_7$ is often designated by the symbol $\sum_{i=3}^{7} n_i$. $\sum$ is the capital Greek letter sigma, and in this connection it is often called the *summation sign*. The letter i is called the *summation index*. The term following $\sum$ is called the *summand*. The "$i = 3$" below $\sum$ indicates that the first term of the sum is obtained by putting $i = 3$ in the summand. The "7" above the $\sum$ indicates that the final term of the sum is obtained by putting $i = 7$ in the summand. The other terms of the sum are obtained by giving i the integral values between the limits 3 and 7. Thus

$$\sum_{j=2}^{5} (-1)^{j-2} j x^{2j} = 2x^4 - 3x^6 + 4x^8 - 5x^{10}.$$

An analogous notation for a product is obtained by substituting the capital Greek letter $\prod$ for $\sum$. In this case the terms resulting from substituting the integers for the index are multiplied instead of added. Thus

$$\prod_{i=1}^{5} \left[a + (-1)^i \frac{i}{b}\right] = \left(a - \frac{1}{b}\right)\left(a + \frac{2}{b}\right)\left(a - \frac{3}{b}\right)\left(a + \frac{4}{b}\right)\left(a - \frac{5}{b}\right).$$

EXAMPLE 1 Some useful formulas involving summations are listed below. They can be proved using mathematical induction.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{1}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2. \tag{3}$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}. \tag{4}$$

Equation (1) can be used to derive the following formula for an *arithmetic series* or *progression:*

$$\sum_{j=1}^{n} [a + (j-1)d] = na + \frac{d}{2} n(n-1). \tag{5}$$

A companion series, the finite *geometric series*, or *progression*, is given by

$$\sum_{j=0}^{n-1} ar^j = a \frac{1-r^n}{1-r}. \tag{6}$$

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2.2 Factorial and Combinatorial Symbols and Conventions

A product of a positive integer n by all the positive integers smaller than it is usually denoted by $n!$ (read "*n factorial*"). Thus

$$n! = n(n-1)(n-2) \cdot \cdots \cdot 1 = \prod_{j=0}^{n-1} (n-j). \tag{7}$$

$0!$ is defined to be 1.

A product of a positive integer n by the next $k-1$ smaller positive integers is usually denoted by $(n)_k$. Thus

$$(n)_k = n(n-1)\cdot\cdots\cdot(n-k+1)$$
$$= \prod_{j=1}^{k} (n-j+1). \tag{8}$$

Note that there are k terms in the product in Eq. (8).

Remark $(n)_k = n!/(n-k)!$, and $(n)_n = n!/0! = n!$. The *combinatorial* symbol $\binom{n}{k}$ is defined as follows:

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{(n-k)!k!}. \tag{9}$$

$\binom{n}{k}$ is read "combination of n things taking k at a time" or more briefly as "n pick k"; it is also called a *binomial coefficient*. Define

$$\binom{n}{k} = 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n. \tag{10}$$

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Remark

$$\binom{n}{0} = \binom{n}{n} = 1.$$

$$\binom{n}{k} = \binom{n}{n-k}.$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \qquad \text{for } n = 1, 2, \ldots \quad \text{and} \quad k = 0, \pm 1, \pm 2, \ldots. \tag{11}$$

Equation (11) is a useful recurrent formula that is easily proved. ////

Both $(n)_k$ and the combinatorial symbol $\binom{n}{k}$ can be generalized from a positive integer n to *any real number* t by defining

$$(t)_k = t(t-1)\cdot\cdots\cdot(t-k+1), \quad \binom{t}{k} = \frac{t(t-1)\cdot\cdots\cdot(t-k+1)}{k!} \qquad \text{for } k = 1, 2, \ldots, \tag{12}$$

and $\binom{t}{k} = 1$ for $k = 0$.

Remark

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdot\cdots\cdot(-n-k+1)}{k!}$$
$$= (-1)^k \frac{n(n+1)\cdot\cdots\cdot(n+k-1)}{k!}$$
$$= (-1)^k \binom{n+k-1}{k}. \qquad ////$$

2.3 Stirling's Formula

In finding numerical values of probabilities, one is often confronted with the evaluation of long factorial expressions which can be troublesome to compute by direct multiplication. Much labor may be saved by using *Stirling's formula*, which gives an approximate value of $n!$. Stirling's formula is

$$n! \approx (2\pi)^{\frac{1}{2}} e^{-n} n^{n+\frac{1}{2}} \tag{13}$$

or

$$n! = (2\pi)^{\frac{1}{2}} e^{-n} n^{n+\frac{1}{2}} e^{r(n)/12n}, \tag{14}$$

where $1 - 1/(12n+1) < r(n) < 1$. To indicate the accuracy of Stirling's formula, 10! was evaluated using five-place logarithms and Eq. (13), and 3,599,000 was obtained. The actual value of 10! is 3,628,800. The percent error is less than 1 percent, and the percent error will decrease as n increases.

2.4 The Binomial and Multinomial Theorems

The *binomial theorem* is often given as

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} \tag{15}$$

for n, a positive integer. The binomial theorem explains why the $\binom{n}{j}$ are sometimes called binomial coefficients. Four special cases are noted in the following remark.

Remark

$$(1+t)^n = \sum_{j=0}^{n} \binom{n}{j} t^j, \tag{16}$$

$$(1-t)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j t^j, \tag{17}$$

$$2^n = \sum_{j=0}^{n} \binom{n}{j}, \tag{18}$$

and

$$0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j}. \qquad (19)$$

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Expanding both sides of

$$(1+x)^a(1+x)^b = (1+x)^{a+b}$$

and then equating coefficients of x to the nth power gives

$$\sum_{j=0}^{n} \binom{a}{j}\binom{b}{n-j} = \binom{a+b}{n}, \qquad (20)$$

a formula that is particularly useful in considerations of the hypergeometric distribution.

A generalization of the binomial theorem is the *multinomial theorem*, which is

$$\left(\sum_{j=1}^{k} a_j\right)^n = \sum \frac{n!}{\prod_{i=1}^{k} n_i!} \prod_{i=1}^{k} a_i^{n_i}, \qquad (21)$$

where the summation is over all nonnegative integers $n_1, n_2, \ldots, n_k$ which sum to n. A special case is

$$\left(\sum_{j=1}^{k} a_j\right)^2 = \left(\sum_{i=1}^{k} a_i\right)\left(\sum_{j=1}^{k} a_j\right) = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j. \qquad (22)$$

Also note that

$$\left(\sum_{i=1}^{m} a_i\right)\left(\sum_{j=1}^{n} b_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j. \qquad (23)$$

3 CALCULUS

3.1 Preliminaries

It is assumed that the reader is familiar with the concepts of limits, continuity, differentiation, integration, and infinite series. A particular limit that is referred to several times in the book is the limit expression for the number e; that is,

$$\lim_{x \to 0} (1+x)^{1/x} = e. \qquad (24)$$

Equation (24) can be derived by taking logarithms and utilizing l'Hospital's rule, which is reviewed below. There are a number of variations of Eq. (24), for instance,

$$\lim_{x\to\infty}(1+x^{-1})^x = e \tag{25}$$

and

$$\lim_{x\to 0}(1+\lambda x)^{1/x} = e^{\lambda} \qquad \text{for constant } \lambda. \tag{26}$$

A rule that is often useful in finding limits is the following so-called *l'Hospital's rule:* If $f(\cdot)$ and $g(\cdot)$ are functions for which $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and if

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

exists, then so does

$$\lim_{x\to a}\frac{f(x)}{g(x)},$$

and

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

EXAMPLE 2 Find $\lim_{x\to 0}[(1/x)\log_e(1+x)]$. Let $f(x) = \log_e(1+x)$ and $g(x) = x$; then

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{1}{1+x} = 1 = \lim_{x\to 0}\left[\frac{1}{x}\log_e(1+x)\right]. \qquad ////$$

Another rule that we use in the book is *Leibniz' rule for differentiating an integral:* Let

$$I(t) = \int_{g(t)}^{h(t)} f(x;t)\,dx,$$

where $f(\cdot\,;\cdot)$, $g(\cdot)$, and $h(\cdot)$ are assumed differentiable. Then

$$\frac{dI}{dt} = \int_{g(t)}^{h(t)}\frac{\partial f}{\partial t}\,dx + f(h(t);t)\frac{dh}{dt} - f(g(t);t)\frac{dg}{dt}. \tag{27}$$

Several important special cases derive from Leibniz' rule; for example, if the integrand $f(x;t)$ does not depend on t, then

$$\frac{d}{dt}\left[\int_{g(t)}^{h(t)} f(x)\,dx\right] = f(h(t))\frac{dh}{dt} - f(g(t))\frac{dg}{dt}; \tag{28}$$

in particular, if $g(t)$ is constant and $h(t) = t$, Eq. (28) simplifies to

$$\frac{d}{dt}\left[\int_c^t f(x)\,dx\right] = f(t). \tag{29}$$

3.2 Taylor Series

The *Taylor series* for $f(x)$ about $x = a$ is defined as

$$f(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n, \qquad (30)$$

where

$$f^{(i)}(a) = \left.\frac{d^i f(x)}{dx^i}\right|_{x=a}; \quad R_n = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} \qquad \text{and} \qquad a \le c \le x.$$

R_n is called the *remainder*. $f(x)$ is assumed to have derivatives of at least order $n + 1$. If the remainder is not too large, Eq. (30) gives a polynomial (of degree n) approximation, when R_n is dropped, of the function $f(\cdot)$. The infinite series corresponding to Eq. (30) will converge in some interval if $\lim_{n\to\infty} R_n = 0$ in this interval. Several important infinite Taylor series, along with their intervals of convergence, are given in the following examples.

EXAMPLE 3 Suppose $f(x) = e^x$ and $a = 0$. Then

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= \sum_{j=0}^{\infty} \frac{x^j}{j!} \qquad \text{for } -\infty < x < \infty. \qquad (31)$$

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EXAMPLE 4 Suppose $f(x) = (1-x)^t$ and $a = 0$; then $f^{(1)}(x) = -t(1-x)^{t-1}$, $f^{(2)}(x) = t(t-1)(1-x)^{t-2}, \ldots, f^{(j)}(x) = (-1)^j t(t-1)\cdot\cdots\cdot(t-j+1)(1-x)^{t-j}$, and hence

$$f(x) = (1-x)^t = \sum_{j=0}^{\infty} (-1)^j (t)_j \frac{x^j}{j!}$$
$$= \sum_{j=0}^{\infty} \binom{t}{j}(-x)^j \qquad \text{for } -1 < x < 1. \qquad (32)$$

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There are several interesting special cases of Eq. (32). $t = -n$ gives

$$(1-x)^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j}(-x)^j = \sum_{j=0}^{\infty} \binom{n+j-1}{j} x^j \qquad \text{for } -1 < x < 1; \qquad (33)$$

$t = -1$ gives the geometric series

$$(1-x)^{-1} = \sum_{j=0}^{\infty} x^j; \qquad (34)$$

$t = -2$ gives

$$(1 - x)^{-2} = \sum_{j=0}^{\infty} (j + 1)x^j. \tag{35}$$

EXAMPLE 5 Suppose $f(x) = \log_e (1 + x)$ and $a = 0$; then

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } -1 < x \leq 1. \tag{36}$$

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The Taylor series for functions of one variable given in Eq. (30) can be generalized to the Taylor series for functions of several variables. For example, the Taylor series for $f(x, y)$ about $x = a$ and $y = b$ can be written as

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) +$$
$$\frac{1}{2!} [f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2] + \cdots,$$

where

$$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{x=a, y=b},$$

$$f_{xy}(a, b) = \left.\frac{\partial^2 f}{\partial y \, \partial x}\right|_{x=a, y=b},$$

and similarly for the others.

3.3 The Gamma and Beta Functions

The *gamma function*, denoted by $\Gamma(\cdot)$, is defined by

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} \, dx \qquad \text{for } t > 0. \tag{37}$$

$\Gamma(t)$ is nothing more than a notation for the definite integral that appears on the right-hand side of Eq. (37). Integration by parts yields

$$\Gamma(t + 1) = t\Gamma(t), \tag{38}$$

and, hence, if $t = n$ (an integer),

$$\Gamma(n + 1) = n!. \tag{39}$$

If n is an integer,

$$\Gamma(n + \tfrac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2^n} \sqrt{\pi}, \tag{40}$$

and, in particular,

$$\Gamma(\tfrac{1}{2}) = 2\Gamma(\tfrac{3}{2}) = \sqrt{\pi}. \tag{41}$$

The *beta function*, denoted by $B(\cdot, \cdot)$, is defined by

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx \qquad \text{for } a > 0, b > 0. \tag{42}$$

Again, $B(a, b)$ is just a notation for the definite integral that appears on the right-hand side of Eq. (42). A simple variable substitution gives $B(a, b) = B(b, a)$. The beta function is related to the gamma function according to the following formula:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{43}$$